Note on Determinants and Duadic Disynthemes.

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A GENERAL algebraical determinant in its developed form (viewed in relation to any one arbitrarily selected term) may be likened to a mixture of liquids seemingly homogeneous, but which being of differing boiling points, admit of being separated by the process of fractional distillation. Thus *ex. gr.* suppose a general determinant of the 6th order. The 720 terms which make it up will fall, in relation to the leading diagonal product, into as many classes (most of which comprise several similarly constituted families) as there are unlimited partitions of 6. These, 11 in number, are

$$6;\ 5, 1;\ 4, 2;\ 4, 1, 1;\ 3, 3;\ 3, 2, 1;\ 3, 1, 1, 1;\ 2, 2, 2;\ 2, 2, 1, 1;\ 2, 1, 1, 1, 1;\ 1, 1, 1, 1, 1, 1.$$

Let the determinant be represented, in the umbral notation, by

$$\begin{matrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{matrix}.\text{*}$$

Let us, by way of illustration, consider the class corresponding to 6; this will consist of the $1.2.3.4.5$ (120) terms obtained by forming the 120 distinct circular arrangements that belong to $a\ b\ c\ d\ e\ f$. Thus:

$$\begin{array}{cccc} & \longrightarrow & & \\ & a \quad c & & \\ b & & & e \\ & f \quad d & & \\ & \longleftarrow & & \end{array}$$

will signify $ac \times ce \times ed \times df \times fb \times ba$, which will be one of the 120 in question. So, again, 3, 3 will denote, in the first place, the 10 sets of double triads of the general form $abc : def$, and, as each triad will give two cyclical orders, there will in all be 10×2^2, *i. e.* 40, terms of the form $ab \,.\, bc \,.\, ca \,.\, de \,.\, ef \,.\, fd$. So, again, there will be 15.1^3, *i. e.* 15, corresponding to 2, 2, 2. So 3, 2, 1 will give 10 groupings of the form $abc : de : f$, and each of these will give rise to two

* The cyclical method of the text shows what was not previously apparent, that the umbral notation $\begin{matrix} ab \ldots l \\ ab \ldots l \end{matrix}$ possesses an essential advantage over $\begin{matrix} ab \ldots l \\ \alpha\beta \ldots \lambda \end{matrix}$ even for unsymmetrical determinants. This mode of notation of course implies some ground of preference for one diagonal group over all others and thus virtually regards a general determinant as related to a lineo-linear as a symmetrical one is to a quadratic form. For instance the general determinant of the second order is to be regarded as appurtenant to the lineo-linear form $aaxx' + abxy' + bayx' + bbyy'$.

terms, viz: $ab.bc.ca.de.ed.ff$, $ac.cb.ba.de.ed.ff$, the number of cycles corresponding to two elements de being 1, and to one element f also 1.

This simple theory affords us a direct means of calculating the number of distinct terms in a symmetrical determinant, *i. e.* one in which $i.j$ and $j.i$ are identical. It enables us to see at once that the coefficient of every term is unity or a power of 2; the rule being that plus or minus terms* of the class corresponding to $m_1, m_2, m_3, \ldots$ will take the coefficient 2^ν, ν being the number of the quantities m which are neither 1 nor 2, for, in every other case, the total number of cycles in each partial group will arrange themselves in pairs which give the same result, thus *ex. gr.*

$$\begin{array}{ccc} & a & \\ d & & b \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ b & & d \\ & c & \end{array}$$

will give the equal products $ab.bc.cd.da$ and $ad.dc.cb.ba$.

As an example of the direct method of computation, take a symmetrical determinant of the 5th order. Write

$$5 \quad 4.1 \quad 3.2 \quad 3.1\,1 \quad 2.2.1 \quad 2\,1.1.1 \quad 1.1.1.1.1.$$

To these 7 classes there will belong respectively

1.12	with the	coefficient	2
5.3	"	"	2
10.1	"	"	2
10.1	"	"	2
15	"	"	1
10	"	"	1
1	"	"	1.

Thus the number of distinct terms will be

$$12 + 15 + 10 + 10 + 15 + 10 + 1 = 73,$$

and the sum of the coefficients

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 120,$$

both of which are right.

Again, if we have a skew determinant of an even order, it will easily be seen that any partition embracing one or more odd numbers will give rise to pairs of terms that mutually cancel, but when all the parts into which the exponent of the order is divided are even, the coefficient will be given by the same rule as for symmetrical determinants, *i. e.* its arithmetical value will be 2^ν, where ν is the number of parts exceeding 2. Thus *ex. gr.* for a skew determinant of the order 6 we have

$$6 \quad 4.2 \quad 2.2.2.$$

*The complete value of the coefficient is $(-)^\mu 2^\nu$, ν being the number of elements in the partition other than 1 or 2, and μ the number of even elements.

The number of terms corresponding to these partitions being 60 with coefficient 2; 15×3 also with coefficient 2, and 15 with coefficient 1, making 120 distinct terms in all, the sum of the coefficients will be

$$120 + 90 + 15 = (1.3.5)^2,$$

which is right, because the result is the square of the sum of 15 synthemes of the form $1.2 \times 3.4 \times 5.6$. It may be observed that 120 is $\frac{15.16}{2}$, as it ought to be, because, until we reach the order 8, the same *double duadic syntheme* can only be made up in one way of two simple ones, but this ceases to be the case from and after 8. Thus *ex. gr.* the pair of synthemes

$$1.2 \quad 3.4 \quad 5.6 \quad 7.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.7 \quad 6.8$$

combined will produce the same double syntheme as the pair

$$1.2 \quad 3.4 \quad 5.7 \quad 6.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.6 \quad 7.8,$$

and accordingly for 8 we have the partitions

$$8 \quad 6.2 \quad 4.4 \quad 4.2.2 \quad 2.2.2.2,$$

giving rise to

2520	with	coefficient	2
28.60	"	"	2
35.3^2	"	"	4
210.3	"	"	2
105	"	"	1,

making in all $2520 + 1680 + 315 + 630 + 105$, *i. e.* 5250, distinct terms, whereas,

$$\frac{(1.3.5.7)^2 + (1.3.5.7)}{2} = 5565,$$

the difference, 315, being due to the fact that there are that number of double synthemes which admit of a twofold resolution into two single synthemes.

I will not stop to prove, but any person conversant with the subject will see at once that this method gives an intuitive and direct proof of the theorem that a pure skew determinant for an even order is a perfect square.* Having only a limited space at my command, I will pass on at once to forming the equation in differences for the case of a symmetrical, a skew, and one or two other special forms of determinants.

1°. For a symmetrical determinant, taking as a diagram, to fix the ideas, the matrix of the 6th order

$$\begin{matrix} a & b & c & d & e & f \\ b & g & h & k & l & m \\ c & h & n & p & q & r \\ d & k & p & s & t & u \\ e & l & q & t & v & w \\ f & m & r & u & w & \omega \end{matrix},$$

* That a skew determinant of an odd order vanishes is apparent from the fact that an odd number cannot be made up of a set of even ones. I use the term skew determinant in its strict sense as referring to a matrix for which $ij = -ji$ and $ii = 0$.

calling u_m the number of distinct terms in a symmetrical matrix of the mth order, and, resolving the entire determinant into a sum of determinants of the order $(m-1)$ multiplied by the letters in the top line, we shall obviously get u_{m-1} together with $(m-1)$ quantities, positive or negative (and we know, by what precedes, that there can be no canceling, so that the sign, for the object in view, may be entirely neglected) of the form

$$\begin{array}{ccccc} b & h & k & l & m \\ c & n & p & q & r \\ b \times d & p & s & t & u\,. \\ e & q & t & v & w \\ f & r & u & w & \omega \end{array}$$

Among these $(m-1)$ quantities all the terms containing bc, bd, be, bf will occur twice over, but those containing b^2 do not recur. Hence, to find the number of distinct terms we may reckon each of such distinct terms as contain bc, bd, be, bf worth only $\frac{1}{2}$, the others counting as 1. But if, instead of the column (which I write as a line) $bcdef$, we had the column $bhklm$, the rule for calculating the number of distinct terms might be calculated by this very same rule, except that the terms multiplied by hc, kd, le, mf ought to count as *units* instead of *halves*. Hence obviously

$$u_m + (m-1)(m-2)\, u_{m-3} \times \frac{1}{2} = u_{m-1} + (m-1)\, u_{m-1} = m u_{m-1},$$

or

$$u_m = m u_{m-1} - \frac{(m-1)(m-2)}{2}\, u_{m-3},$$

which is Mr. Cayley's equation, but obtained by a much more expeditious process (see Salmon's Higher Algebra, 3d edition, pp. 40–42); writing $u_m = (1 \,.\, 2 \ldots m)\, v_m$ we obtain the equation in differences, linear in regard to the independent variable,

$$m v_m - m v_{m-1} + \frac{v_{m-3}}{2} = 0,$$

and this, treated by the general method applicable to all such, gives rise to a linear differential equation in which, on account of the particular initial values of u_0, u_1, u_2, the third term is wanting, and finally v_m is found to be the coefficient of t^m in

$$\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}},$$

If we apply a similar method to the case of a symmetrical determinant in which the diagonal of symmetry is filled out with zeros (an invertebrate symmetrical or symmetrical bialar determinant, as we may call it) we shall easily obtain the equation in differences

$$u_m = (m-1)\,[u_{m-1} + u_{m-2}] - \frac{(m-1)(m-2)}{2}\,u_{m-3},$$

and, making $u_m = 1.2\ldots mv_m$,

$$mv_m - (m-1)\,v_{m-1} - v_{m-2} + \frac{u_{m-3}}{2} = 0,$$

from which, calling $y = v_0 + v_1 t + v_2 t^2 + \ldots$ and having regard to the initial values v_0, v_1, v_2, we obtain

$$2\,\frac{dy}{y} = \frac{2t - t^2}{1-t}\,dt,$$

and

$$y = \frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}}.$$

By way of distinction, using u' and v' for this case, and u, v for the preceding one, the slightest consideration shows that

$$u_m = u'_m + mu'_{m-1} + \frac{m(m-1)}{2}\,u'_{m-2} + \frac{m(m-1)(m-2)}{2.3}\,u'_{m-3} + \ldots,$$

or

$$v_m = v'_m + v'_{m-1} + \frac{v'_{m-2}}{1.2} + \frac{v'_{m-3}}{1.2.3} + \ldots.$$

Hence the generating function for v_m ought to be that for u_m multiplied by e^t, as we see is the case.

So, in like manner, the generating function for v_m, *i. e.* $\frac{u_m}{1.2\ldots m}$, in the case of a general determinant being $\frac{1}{1-t}$, that of v_m for an invertebrate or zero-axial but otherwise general determinant we see must be $\frac{e^{-t}}{1-t}$*, *i. e.*

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*It may easily be proved that the difference between the numbers of positive and negative combinations in the development of an invertebrate determinant of the mth order is $(-)^{m-1}(m-1)$ in favor of the former. From this it is easy to prove that the generating function for $\frac{\text{number of positive terms in such determinant}}{1.2.3\ldots m}$ is

$$\frac{1}{2}\left\{\frac{e^{-t}}{1-t} - (1+t)\,e^{-t}\right\}, \text{ or } \frac{t^2 e^{-t}}{2(1-t)}.$$

Whence it follows that the number of positive terms in a general invertebrate determinant of the mth order is $m\,\frac{m-1}{2}$ times the total number of the terms in one of the $(m-2)$th order. The equation of differences for U_m, the total number, is of course

$$U_m = (m-1)(U_{m-1} + U_{m-2}),$$

and the successive values of

U_m for	1,	2,	3,	4,	5,	6,	7,	8,	. . . .
are	0,	1,	2,	9,	44,	265,	1854,	14833,	. . . .

$$v_m = 1 - 1 + \frac{1}{1.2} - \frac{1}{1.2.3} + \ldots \pm \frac{1}{1.2 \ldots . m},$$

the well known value $\left(\text{ultimately equal to } \frac{1}{e}\right)$, as it ought obviously to be, of the chance of two cards of the same name not coming together when one pack of m distinct cards is laid card for card under another precisely similar pack.

Returning to the case of the invertebrate symmetrical determinant, it will readily be seen, by virtue of the prolegomena, that the number of terms (the u_m) for such a determinant of the mth order is the same thing as the total number of duadic disynthemes that can be formed with m things, meaning by a duadic disyntheme any combination of duads with or without repetition, in which each element occurs twice and no oftener. Thus, when $m = 6$, $1.2 \ 2.3 \ 1.3 \ 4.5 \ 4.6 \ 5.6$ and $1.2 \ 2.3 \ 3.4 \ 5.6 \ 6.1$ and $1.2 \ 2.3 \ 3.4 \ 1.4 \ 5.6 \ 5.6$ are all three of them disynthemes. But the two latter ones are each resoluble into single synthemes, whereas the first one is not. It is clear that, when a disyntheme is formed by means of cycles all of an even order, it will be resoluble into a pair of single synthemes, and in no other case. The problem, then, of finding the number of distinct double synthemes with m elements is one and the same as that of finding the number of distinct terms in a *proper* (*i. e.* invertebrate) skew determinant, which I proceed to consider.

Following a method (not identical with but) analogous to that adopted for the symmetrical cases, we shall find, by a process which the terms below written will sufficiently suggest

$$u_m + \frac{(m-1)(m-2)(m-3)}{2} u_{m-4} = (m-1)\, u_{m-2} + (m-1)(m-2)\, u_{m-2},$$

or

$$u_m = (m-1)^2\, u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2} u_{m-4}.$$

Of course, when m is odd $u_m = 0$. From this it is readily seen that $\frac{u_{2m}}{1.3.5 \ldots 2_{m-1}}$, say ω_m, is an integer; for we shall have

$$\omega_m = (2m-1)\, \omega_{m-1} - (m-1)\, \omega_{m-2},$$

also,

$$\omega_1 = 1, \quad \omega_2 = 2,$$

so that

$$\begin{aligned} \omega_3 &= 5.2 - 2.1 = 8, \\ \omega_4 &= 7.8 - 3.2 = 50, \\ \omega_5 &= 9.50 - 4.8 = 418, \\ \omega_6 &= 11.418 - 5.50 = 4348, \end{aligned}$$

and the conventional $\omega_0 = 3\omega_1 - \omega_2 = 1$.

By the above formula u_m can be calculated with prodigious rapidity. If, however, we wish to obtain a generating function for u_m, the differential equation obtained from the above equation in differences does not lead to a simple explicit integral, but if we make $u_{2m} = (1.2.3\ldots 2m)\,v_m$, as in the preceding cases, or, which is the same thing, $\omega_m = 2_m\,(1.2\ldots m)\,v_m$, we get

$$4mv_m - 4\,(m-1)\,v_{m-1} - 2v_{m-1} + v_{m-2} = 0\,,$$

and, writing as before $\quad y = v_0 + v_1 t + v_2 t^2 + \ldots,$

$$4\,\frac{dy}{dt} - 4t\,\frac{dy}{dt} - 2y + ty$$

will be found to be equal to zero. [This vanishing of the 3d term in the differential equation being a feature common to all the cases we have considered, and due to the initial values of the v series in each case.] We have thus

$$\frac{4y'}{y} = \frac{1}{1-t} + 1\,, \quad y = \frac{e^{\frac{t}{4}}}{(1-t)^{\frac{1}{4}}}\,.$$

By way of verification, we may observe that

$$v_0 = 1\,, \quad v_1 = \frac{1}{2}\,, \quad v_2 = \frac{1}{4}\,, \quad v_3 = \frac{1}{6}\,, \ldots,$$

$$y = \left(1 + \frac{t}{4} + \frac{t^2}{32} + \frac{t^3}{384} + \ldots\right)\left(1 + \frac{t}{4} + \frac{5t^2}{32} + \frac{45t^3}{384} + \ldots\right),$$

and

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}\,, \quad \frac{1}{32} + \frac{1}{16} + \frac{5}{32} = \frac{1}{4}\,, \quad \frac{45}{384} + \frac{5}{128} + \frac{1}{128} + \frac{1}{384} = \frac{1}{6}\text{*}.$$

We may now proceed to calculate the number of distinct terms in an improper or vertebrated skew-determinant, which is interesting on account of its connection with the theory of orthogonal transformations. Using v_{2m}, instead of v_m, the generating function for the case last considered becomes $\dfrac{e^{\frac{t^2}{4}}}{\sqrt[4]{1-t^2}}$. Let $(1.2.3\ldots m)\,V_m = U_m$ in general be used to denote the number of distinct terms in a vertebrate skew-determinant of the mth order. Then obviously

$$U_{2m} = u_{2m} + m\,.\,\frac{m-1}{2}\,u_{2m-2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{m-3}{4}\,u_{2m-4} + \ldots,$$

or

$$V_{2m} = v_{2m} + \frac{v_{2m-2}}{1.2} + \frac{v_{2m-4}}{1.2.3.4} + \ldots.$$

Hence the generating function for V_{2m}

$$= \frac{e^{\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\left\{1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \ldots\right\} = \frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} + e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\},$$

* The values of $v_1, v_2, v_3 \ldots$ are $\frac{1}{2}, \frac{2}{2.4}, \frac{8}{2.4.6}, \frac{50}{2.4.6.8}, \ldots$; *i. e.* $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{25}{192}, \ldots$

and in like manner, since

$$U_{2m-1} = mu_{2m-2} + \frac{m(m-1)(m-2)}{1.2.3} u_{2m-4} + \ldots,$$

the generating function for V_{2m-1} will be

$$\frac{1}{2} \left\{ \frac{e^{t+\frac{t^2}{4}} - e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}} \right\}.$$

Hence the number of distinct cross-products in the development of an orthogonal transformation-matrix of the mth order is

$$(1.2.3\ldots m) \times \text{coefficient of } t^m \text{ in } \frac{e^{t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}.$$

POSTSCRIPT.—Let us consider the case of $2m$ elements; call the number of ways in which any disyntheme composed with them may be resolved into a pair of single synthemes one in each hand* its weight; furthermore, call the aggregate of those which appertain to an odd number of cycles the first class, and the other the second class. The entire sum of the weights we know is $1^2.2^2.3^2\ldots\overline{2m-1}^2$, but, furthermore, I find that the excess of the total weight of the first class over that of the second is $1^2.2^2.3^2\ldots\overline{2m-3}^2.\overline{2m-1}$; or, in other words, the weights of the two classes are in the ratio of m to $m-1$.

The expressions for the sum and for the difference may, of course, by the *prolegomena* be translated into two theorems on the partition of numbers, neither of which, as far as I can see, is obvious upon the face of it.†

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* The two hands are introduced in order to double, by the effect of permutation, what the weight otherwise would be, except when the two component synthemes are identical, in which case the weight remains unity.

† REMARK.—The equation in differences for the number of double duadic synthemes may be obtained without recourse to determinants, as follows: Single out any element, 1; it may be paired in each of the component synthemes with any one of the remaining elements 2, 3, 4, ..., and there are two cases to be distinguished, viz: 1 may be paired either with the same element (2) or with two different elements (2, 3), in the two synthemes. The former may be done in $(m-1)$ ways, and, after having made our choice, we have still the choice of all the double synthemes that can be formed from $3, 4, \ldots m$; $3, 4, \ldots m$. The choice of two *different* elements may be made in $\frac{(m-1)(m-2)}{2}$ ways, and having chosen, we have still the choice of all the double synthemes that can be formed from $3, 4, \ldots m$; $2, 4, \ldots m$. Now it is plain that the number of these can be obtained from the number of double synthemes that can be formed from $3, 4, \ldots m$; $3, 4, \ldots m$, by counting twice all except those in which 3 is paired twice with the same element; and is equal, therefore, from what precedes, to

$$2u_{m-2} - (m-3)\, u_{m-4}.$$

We have, therefore,

$$\begin{aligned} u_m &= (m-1)\, u_{m-2} + \frac{(m-1)(m-2)}{2} [2u_{m-2} - (m-3)\, u_{m-4}] \\ &= (m-1)^2\, u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2} u_{m-4}. \end{aligned}$$

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